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Nonisospectral integrable nonlinear equations with external potentials and their GBDT solutions

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Abstract

Auxiliary systems for matrix nonisospectral equations, including coupled NLS with external potential and KdV with variable coefficients, were introduced. Explicit (up to the inversion of finite matrices) solutions of nonisospectral equations were constructed using the GBDT version of the Bäcklund–Darboux transformation.

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1. Introduction

The nonisospectral method to generate solvable nonlinear evolution equations was proposed in the seminal paper [6] and actively used in a wide range of papers (see, for instance, various references in the recent papers [22, 38, 39]). In particular, this approach allows us to generate physically interesting nonlinear equations with external potentials [7, 8]. Such equations are used to describe deep water and plasma waves, waves in non-uniform media, light pulses and energy transport (see [10, 35, 38] and references therein).

In the paper we assume that the spectral parameter λ depends on both variables x and t . Important results for this case were obtained by Burtsev, Zakharov and Mikhailov in [5] (see also interesting papers [3, 4, 21]). Interesting deformations of the KdV, MKdV and nonlinear Schrödinger equations were constructed in [5]. Physically important nonisospectral deformations of the Maxwell–Bloch system were constructed and studied in [3–5] too. A general dressing method for the case of ‘moving poles’ was proposed in [5]. In particular, the so-called polynomial case

$$\lambda_x = p_0 + p_1\lambda + \dots + p_k\lambda^k, \quad \lambda_t = \tilde{p}_0 + \tilde{p}_1\lambda + \dots + \tilde{p}_l\lambda^l, \quad 1 \leq k \leq l \quad (1.1)$$

was treated there.

In our paper we consider in detail the polynomial case, where $k = 0$. It proves that this case deserves a separate consideration: we construct some solvable generalizations of the matrix versions of the coupled nonlinear Schrödinger (CNLS), KdV and MKdV equations.

Matrix versions are of interest (see, for instance, [9]) and include, in particular, scalar cases and multi-component cases. We construct auxiliary linear systems for our matrix (and scalar) cases and some equations seem to be new in the scalar cases too.

Afterwards we apply to the constructed equations GBDT, which is a version of the Bäcklund–Darboux transformation developed by the authors in [24–32]. The Bäcklund–Darboux transformation is a well-known and fruitful tool to construct explicit solutions of the integrable equations and some classical linear equations as well. Some important versions of the Bäcklund–Darboux transformation one can find in [11, 12, 15, 17, 19, 20, 37]. The GBDT version of the iterated Bäcklund–Darboux transformation is of rather general nature and provides simple algebraic formulae based on some system theoretical and matrix identity results. The parameter matrices, which are used in GBDT, have an arbitrary Jordan structure, while diagonal matrices (with eigenvalues of the auxiliary spectral problems as the entries) are mostly used in other approaches.

Examples are considered in a more detailed way.

2. Nonisospectral equations

2.1. NLS with external potential

Recall the coupled nonlinear Schrödinger equation (CNLS) of the form [14]:

$$v_{1t} + iv_{1xx} + 2iv_1v_2v_1 = 0, \quad v_{2t} - iv_{2xx} - 2iv_2v_1v_2 = 0 \quad \left(v_t := \frac{\partial}{\partial t} v \right). \quad (2.1)$$

We shall consider the matrix version of (2.1), where v_1 and v_2 are $m_1 \times m_2$ and $m_2 \times m_1$ ($m_1, m_2 \geq 1$) matrix functions, respectively. Auxiliary linear systems for CNLS are given by the formulae

$$w_x(x, t, \lambda) = G(x, t, \lambda)w(x, t, \lambda), \quad w_t(x, t, \lambda) = F(x, t, \lambda)w(x, t, \lambda). \quad (2.2)$$

Here we have

$$G = -(\lambda q_1 + q_0), \quad F = -(\lambda^2 Q_2 + \lambda Q_1 + Q_0), \quad (2.3)$$

where λ is the independent of x and t spectral parameter,

$$2q_1 = -Q_2 = 2ij, \quad 2q_0(x, t) = -Q_1(x, t) = 2j\xi(x, t), \quad (2.4)$$

$$Q_0(x, t) = i(j\xi(x, t)^2 - \xi_x(x, t)), \quad (2.5)$$

$$j = \begin{bmatrix} I_{m_1} & 0 \\ 0 & -I_{m_2} \end{bmatrix}, \quad \xi = \begin{bmatrix} 0 & v_1 \\ v_2 & 0 \end{bmatrix}, \quad (2.6)$$

and I_k is the $k \times k$ identity matrix. We assume in this section that G and F are continuous together with their first derivatives. Then the compatibility condition for systems (2.2) can be written in the zero-curvature equation form:

$$G_t(x, t, \lambda) - F_x(x, t, \lambda) + G(x, t, \lambda)F(x, t, \lambda) - F(x, t, \lambda)G(x, t, \lambda) = 0. \quad (2.7)$$

It can be checked directly that (2.1) is equivalent to the compatibility condition (2.7). In other words, equation (2.1) can be presented in the zero-curvature form (2.7). (See [13] on the historical details about zero-curvature representation of the integrable equations.) It follows from (2.7) that (2.1) can be presented in the form:

$$-j(\xi_t(x, t) + ij\xi_{xx}(x, t) + 2ij\xi(x, t)^3) = 0. \quad (2.8)$$

Indeed, the left-hand side of (2.8) coincides with the left-hand side of (2.7), while the coefficients at the degrees λ^k ($k > 0$) on the left-hand side of (2.7) turn to zero. It is

easy to calculate that in the nonisospectral case (for λ depending on x and t) the zero-curvature equation (2.7) is equivalent to

$$-\lambda_t q_1 + 2\lambda_x \lambda Q_2 + \lambda_x Q_1 - j(\xi_t(x, t) + ij\xi_{xx}(x, t) + 2ij\xi(x, t)^3) = 0. \quad (2.9)$$

Assume now

$$\lambda_t = -4\lambda_x \lambda, \quad (2.10)$$

to derive from (2.4), (2.9) and (2.10)

$$(\xi_t(x, t) + ij\xi_{xx}(x, t) + 2ij\xi(x, t)^3) + 2\lambda_x \xi(x, t) = 0. \quad (2.11)$$

Thus we obtain the following proposition.

Proposition 2.1. *The CNLS with external potential*

$$v_{1t} + iv_{1xx} + 2iv_1 v_2 v_1 + 2\lambda_x v_1 = 0, \quad v_{2t} - iv_{2xx} - 2iv_2 v_1 v_2 + 2\lambda_x v_2 = 0 \quad (2.12)$$

is a nonisospectral integrable equation, in which auxiliary linear systems are given by (2.2)–(2.6), where $\lambda_t = -4\lambda_x \lambda$. In other words, equation (2.12) admits zero-curvature representation (2.7), where G and F are given by (2.3)–(2.6) and $\lambda_t = -4\lambda_x \lambda$.

Remark 2.2. The simplest solution of (2.11) is given by the function

$$\lambda(x, t) = \frac{1}{4}(x + c)(t + b)^{-1}, \quad (2.13)$$

where c is the ‘hidden spectral parameter’ in the terminology of [5]. Correspondingly, we obtain an integrable coupled nonlinear Schrödinger equation with a simple external potential

$$v_{1t} + iv_{1xx} + 2iv_1 v_2 v_1 + \frac{1}{2(t + b)} v_1 = 0, \quad v_{2t} - iv_{2xx} - 2iv_2 v_1 v_2 + \frac{1}{2(t + b)} v_2 = 0. \quad (2.14)$$

The problem of similarity transformations [18, 23] is of interest here. When $b = \bar{b}$, using $\lambda(x, t)$ as in (2.13) one can construct an integrable nonlinear Schrödinger equation (NLS) with external potential [31], however the substitution

$$t = -b - \tilde{t}^{-1}, \quad x = -\tilde{x}\tilde{t}^{-1} \quad (2.15)$$

turns it (see [23], section 6, example 4) into the classical cubic NLS. In a quite similar way substitution (2.15) and equalities

$$\tilde{v}_1 = (t + b) \exp\left(\frac{ix^2}{4(t + b)}\right) v_1, \quad \tilde{v}_2 = (t + b) \exp\left(-\frac{ix^2}{4(t + b)}\right) v_2 \quad (2.16)$$

transform (2.14) in the case $b = \bar{b}$ into the CNLS

$$\tilde{v}_{1\tilde{t}} + i\tilde{v}_{1\tilde{x}\tilde{x}} + 2i\tilde{v}_1 \tilde{v}_2 \tilde{v}_1 = 0, \quad \tilde{v}_{2\tilde{t}} - i\tilde{v}_{2\tilde{x}\tilde{x}} - 2i\tilde{v}_2 \tilde{v}_1 \tilde{v}_2 = 0. \quad (2.17)$$

The case $b \neq \bar{b}$ is more interesting from that point of view, though GBDT for equation (2.14) ($b = \bar{b}$) can be applied to construct new solutions of (2.1) and proves therefore useful too.

2.2. KdV equation with variable coefficients

The matrix KdV equation can be written as

$$v_t(x, t) - 3v(x, t)v_x(x, t) - 3v_x(x, t)v(x, t) + v_{xxx}(x, t) = 0, \quad (2.18)$$

where v is a $p \times p$ matrix function. Equation (2.18) admits zero-curvature representation (2.7), where polynomials G and F of form (2.3) are defined via the $m \times m$ ($m = 2p$) coefficients

$$q_1 = \begin{bmatrix} 0 & 0 \\ I_p & 0 \end{bmatrix}, \quad q_0 = -\begin{bmatrix} 0 & I_p \\ v & 0 \end{bmatrix}, \quad (2.19)$$

$$Q_2 = \begin{bmatrix} 0 & 0 \\ 4I_p & 0 \end{bmatrix}, \quad Q_1 = -\begin{bmatrix} 0 & 4I_p \\ 2v & 0 \end{bmatrix}, \quad Q_0 = \begin{bmatrix} v_x & -2v \\ v_{xx} - 2v^2 & -v_x \end{bmatrix}. \quad (2.20)$$

Now, substitute (2.20) by the equalities

$$\begin{aligned} Q_2 &= g(t) \begin{bmatrix} 0 & 0 \\ 4I_p & 0 \end{bmatrix}, & Q_1 &= -g(t) \begin{bmatrix} 0 & 4I_p \\ 2v & 0 \end{bmatrix}, \\ Q_0 &= g(t) \left(\begin{bmatrix} v_x & -2v \\ v_{xx} - 2v^2 & -v_x \end{bmatrix} + 2f(t)j \right), & j &= \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}, \end{aligned} \quad (2.21)$$

where g and f are scalar functions. One can easily take into account corresponding changes in (2.7) and obtain our next proposition.

Proposition 2.3. (a) Assume $\lambda = xf(t) + h(t)$. Then equation (2.7), where G and F are defined via (2.19) and (2.21), is a zero-curvature representation of KdV with variable coefficients and external potential:

$$v_t + g(t)(v_{xxx} - 3vv_x - 3v_xv - 6fv) = (x(f_t - 12gf^2) + h_t - 12fgh)I_p. \quad (2.22)$$

(b) Assume

$$\lambda_x = f, \quad \lambda_t = 12fg\lambda, \quad f_t = 12gf^2. \quad (2.23)$$

Then equation (2.7), where G and F are defined via (2.19) and (2.21), is a zero-curvature representation of a special case of equation (2.22):

$$v_t + g(t)(v_{xxx} - 3vv_x - 3v_xv - 6fv) = 0. \quad (2.24)$$

Note that equation (2.22), where $p = 1$ (scalar case) and $h = 0$, was treated using the homogeneous balance principle in [36]. When (2.23) holds and $g = 1$, one can put $f = -(t + b)^{-1}/12$. The corresponding equation

$$v_t + v_{xxx} - 3vv_x - 3v_xv + \frac{1}{2(t+b)}v = 0$$

appeared in [1] and its subcase $b = 0$ is a well-known cylindrical KdV [7].

2.3. MKdV with external potential

Introduce G and F by the equalities

$$\begin{aligned} G &= i\lambda j + \xi B, & F &= -i\lambda^3 j - \lambda^2 \xi B - \frac{i\lambda}{2}(\xi_x B j + (\xi B)^2) \\ & & &+ \frac{1}{4}(\xi_{xx} B - (-1)^k(2\xi^3 B + \xi_x \xi - \xi \xi_x)) + \frac{i}{2} \partial_x^{-1}(\lambda_x (\xi B)^2), \end{aligned} \quad (2.25)$$

where j is given in (2.6), $B = j^k$ ($k = 0, 1$), $\lambda = \bar{\lambda} = \sqrt{(x+c)/3(t+b)}$, and

$$\xi = \begin{bmatrix} 0 & -v^* \\ v & 0 \end{bmatrix}. \tag{2.26}$$

Then representation (2.7) is equivalent to the equation

$$4v_t = v_{xxx} + 3(-1)^k(v_x v^* v + v v^* v_x) - \frac{4}{3(t+b)}v - 2i f v_x + 2(-1)^k i(v \partial_x^{-1}(f v^* v) - \partial_x^{-1}(f v v^*)v),$$

$$f(x, t) = (2\sqrt{3(x+c)(t+b)})^{-1}. \tag{2.27}$$

Some other generalizations of MKdV one can find in [2, 38].

3. GBDT for the nonisospectral case: preliminaries

GBDT (nonisospectral case) for systems with rational dependence on the spectral parameter has been introduced in [25]. Here we shall need a reduction of the theorem in [25] (section 2, p 1253) for the first-order systems of the form

$$w'(u, \lambda) = \mathcal{G}(u, \lambda)w(u, \lambda) \quad \left(w' = \frac{d}{du} w \right), \quad \mathcal{G}(u, \lambda) = - \sum_{k=0}^r \lambda^k \mathcal{Q}_k(u), \tag{3.1}$$

where the coefficients $\mathcal{Q}_k(u)$ are $m \times m$ locally summable on the interval $(-c_1, c_2)$ ($c_1, c_2 \geq 0$) matrix functions. It was assumed in [25] that the derivative of $\lambda = \lambda(u)$ rationally depends on λ , but now it will suffice to suppose a polynomial dependence:

$$\lambda'(u) = \sum_{k=0}^r \omega_k(u) \lambda(u)^k. \tag{3.2}$$

After fixing an integer $n > 0$ the GBDT of the system (3.1) is determined by the five parameter matrices: three $n \times n$ matrices $A_1(0)$, $A_2(0)$ and $S(0)$ ($\det S(0) \neq 0$) and two $n \times m$ matrices $\Pi_1(0)$ and $\Pi_2(0)$, such that

$$A_1(0)S(0) - S(0)A_2(0) = \Pi_1(0)\Pi_2(0)^*. \tag{3.3}$$

Next, introduce the matrix functions $A_1(u)$, $A_2(u)$, $\Pi_1(u)$, $\Pi_2(u)$ and $S(u)$ with the given above values at $u = 0$ by the differential equations

$$A_l'(u) = \sum_{k=0}^r \omega_k(u) A_l(u)^k \quad (l = 1, 2), \tag{3.4}$$

$$\Pi_1'(u) = \sum_{k=0}^r A_1(u)^k \Pi_1(u) \mathcal{Q}_k(u), \quad \Pi_2'(u) = - \sum_{k=0}^r (A_2(u)^*)^k \Pi_2(u) \mathcal{Q}_k(u)^*, \tag{3.5}$$

$$S'(u) = \sum_{k=1}^r \sum_{j=1}^k A_1(u)^{k-j} (\Pi_1(u) \mathcal{Q}_k(u) \Pi_2(u)^* - \omega_k(u) S(u)) A_2(u)^{j-1}. \tag{3.6}$$

Note that equations (3.4)–(3.6) are chosen in such a way that the identity

$$A_1(u)S(u) - S(u)A_2(u) = \Pi_1(u)\Pi_2(u)^* \tag{3.7}$$

follows from (3.3) for all u in the connected domain, where the coefficients \mathcal{Q}_k are defined. (The relation is obtained by the direct differentiation of both sides of (3.7).) Assuming that $\det S(u) \neq 0$ and $\det(A_1(u) - \lambda(u)I_n) \neq 0$ we define a transfer matrix function

$$w_A(u, \lambda) = I_m - \Pi_2^* S^{-1} (A_1 - \lambda I_n)^{-1} \Pi_1. \tag{3.8}$$

Remark 3.1. Following the notation in the isospectral case we write $w_A(u, \lambda)$, but as $\lambda = \lambda(u)$, so one can write simply $w_A(u)$.

Transfer matrix functions of the form

$$w_A(\lambda) = I_{\mathcal{L}} - \Pi_2^* S^{-1} (A_1 - \lambda I_{\mathcal{H}})^{-1} \Pi_1 \quad (A_1 S - S A_2 = \Pi_1 \Pi_2^*),$$

where $I_{\mathcal{L}}$ and $I_{\mathcal{H}}$ are the identity matrices in the Hilbert spaces \mathcal{L} and \mathcal{H} , respectively, have been introduced and studied by L Sakhnovich in the context of his method of operator identities (see [33, 34] and references in [34]) and take roots in the Livšic characteristic matrix functions. Our next theorem is a reduction of the theorem from [25].

Theorem 3.2. Let relations (3.2) and (3.3) hold. Define matrix functions A_1, A_2, Π_1, Π_2 and S by the equalities (3.4)–(3.6). Then in the points of invertibility of the matrix functions $S(u)$ and $A_1(u) - \lambda(u)I_n$ the equation

$$w'_A(u, \lambda) = \widehat{\mathcal{G}}(u, \lambda) w_A(u, \lambda) - w_A(u, \lambda) \mathcal{G}(u, \lambda), \quad \widehat{\mathcal{G}}(u, \lambda) = - \sum_{k=0}^r \lambda(u)^k \widehat{\mathcal{Q}}_k(u) \quad (3.9)$$

is true, and the coefficients $\widehat{\mathcal{Q}}_k$ are given by the formulae

$$\begin{aligned} \widehat{\mathcal{Q}}_k = & \mathcal{Q}_k - \sum_{j=k+1}^r \left(\mathcal{Q}_j Y_{j-k-1} - X_{j-k-1} \mathcal{Q}_j + \sum_{s=k+2}^j X_{j-s} \mathcal{Q}_j Y_{s-k-2} \right) \\ & + \sum_{j=k+2}^r \omega_j \sum_{s=k+2}^j Z_{j-s, s-k-2}, \end{aligned} \quad (3.10)$$

where

$$X_k = \Pi_2^* S^{-1} A_1^k \Pi_1, \quad Y_k = \Pi_2^* A_2^k S^{-1} \Pi_1, \quad (3.11)$$

$$Z_{k,j} = \Pi_2^* S^{-1} A_1^k S A_2^j S^{-1} \Pi_1. \quad (3.12)$$

Remark 3.3. According to theorem 3.2 the matrix function w_A is a Darboux matrix, which transforms solution w of system (3.1) into solution $\widehat{w} = w_A w$ of the system $\widehat{w}' = \widehat{\mathcal{G}} \widehat{w}$.

Finally, we shall need also formula (9) from [25]:

$$(\Pi_2^* S^{-1})' = - \sum_{k=0}^r \check{\mathcal{Q}}_k \Pi_2^* S^{-1} A_1^k, \quad \check{\mathcal{Q}}_k := \widehat{\mathcal{Q}}_k - (k+1)\omega_{k+1} I_m \quad (\omega_{r+1} = 0). \quad (3.13)$$

4. Solutions of equations with external potentials

In this section we shall construct GBDT solutions for the CNLS equation (2.14) and for the KdV-type equation (2.24). First, let v_1 and v_2 satisfy (2.14). This means that the zero-curvature equation (2.7), where G and F are defined by (2.3)–(2.6) and

$$\lambda_x = f(t), \quad \lambda_t = -4f(t)\lambda \quad (f(t) = \frac{1}{4}(t+b)^{-1}), \quad (4.1)$$

holds. Now, put $m = m_1 + m_2$ and apply GBDT from section 3 for systems (2.2). Putting $u = x$ and using the notations of (3.2), we derive from (2.3) and (4.1) that $r = 1, \omega_1 = 0, \omega_0(x) \equiv \text{const}$, i.e., $\omega_0(x, t) = f(t)$, where t is a second variable. Putting $u = t$, we

derive $r = 2$, $\omega_2 = \omega_0 = 0$, $\omega_1(t) = -4f(t)$. Hence, formula (3.4) for $A = A_1, A_2$ after substitutions $u = x$ and $u = t$ takes the form

$$A_x = f(t)I_n, \quad A_t = -4f(t)A. \tag{4.2}$$

Therefore we can put

$$A_1(x, t) = f(t)(xI_n + a_1), \quad A_2(x, t) = f(t)(xI_n + a_2), \tag{4.3}$$

where a_1 and a_2 are some arbitrary $n \times n$ matrices. We require (compare with (3.3)):

$$A_1(0, 0)S(0, 0) - S(0, 0)A_2(0, 0) = \Pi_1(0, 0)\Pi_2(0, 0)^*. \tag{4.4}$$

Next, we introduce the matrix functions $\Pi_1(x, t)$ and $\Pi_2(x, t)$, where the dependence on x is determined by the system $w_x = Gw$, that is, by the coefficients of G , and the dependence on t is determined by the system $w_t = Fw$. Namely, when we put $u = x$, the system for Π_1 in (3.5) takes the form

$$(\Pi_1(x, t))_x = A_1(x, t)\Pi_1(x, t)q_1 + \Pi_1(x, t)q_0(x, t), \tag{4.5}$$

and when we put $u = t$, the system takes the form

$$(\Pi_1(x, t))_t = A_1(x, t)^2\Pi_1(x, t)Q_2 + A_1(x, t)\Pi_1(x, t)Q_1(x, t) + \Pi_1(x, t)Q_0(x, t). \tag{4.6}$$

The compatibility of systems (4.5) and (4.6) follows from (2.7). In a similar way we rewrite the second equation in (3.5):

$$(\Pi_2)_x = -\sum_{k=0}^1 (A_2^*)^k \Pi_2 q_k^*, \quad (\Pi_2)_t = -\sum_{k=0}^2 (A_2^*)^k \Pi_2 Q_k^*. \tag{4.7}$$

Recall now relations

$$\begin{aligned} \omega_1 = 0, \quad \omega_0 = f(t), \quad \text{when } u = x; \\ \omega_2 = \omega_0 = 0, \quad \omega_1 = -4f(t), \quad \text{when } u = t. \end{aligned} \tag{4.8}$$

In view of (4.8) we rewrite (3.6) as

$$S_x = \Pi_1 q_1 \Pi_2^*, \quad S_t = \Pi_1 Q_1 \Pi_2^* + A_1 \Pi_1 Q_2 \Pi_2^* + \Pi_1 Q_2 \Pi_2^* A_2 + 4f(t)S. \tag{4.9}$$

According to (4.3)–(4.7) and (4.9) the matrix identity

$$A_1 S - S A_2 \equiv \Pi_1 \Pi_2^* \tag{4.10}$$

is true. Finally, using (3.10) and equality $X_0 = Y_0$, define coefficients $\widehat{q}_k(x, t)$ and $\widehat{Q}_k(x, t)$:

$$\widehat{q}_1 = q_1, \quad \widehat{q}_0 = q_0 + X_0 q_1 - q_1 X_0, \tag{4.11}$$

$$\widehat{Q}_2 = Q_2, \quad \widehat{Q}_1 = Q_1 + X_0 Q_2 - Q_2 X_0,$$

$$\widehat{Q}_0 = Q_0 + X_0 Q_1 - Q_1 X_0 + X_1 Q_2 - Q_2 Y_1 - X_0 Q_2 X_0. \tag{4.12}$$

Partition matrix functions Π_l ($l = 1, 2$): $\Pi_1 = [\Lambda_1 \ \Lambda_2]$ and $\Pi_2 = [\Psi_1 \ \Psi_2]$, where Λ_1, Ψ_1 are $n \times m_1$ blocks, and Λ_2, Ψ_2 are $n \times m_2$ blocks. From theorem 3.2 and remark 3.3 proposition follows.

Proposition 4.1. *Let v_1 and v_2 satisfy CNLS (2.14) with external potential. Then, in the points of invertibility of S and $A_1 - \lambda I_n$, the matrix functions*

$$\widehat{v}_1 = v_1 - 2i\Psi_1^* S^{-1} \Lambda_2, \quad \widehat{v}_2 = v_2 - 2i\Psi_2^* S^{-1} \Lambda_1 \tag{4.13}$$

also satisfy equation (2.14).

Proof. Equations (3.8), (4.3)–(4.7) and (4.9) define the transfer matrix function $w_A(x, t, \lambda)$. By remark 3.3 for $\widehat{w} = w_A w$ we have $\widehat{w}_x = \widehat{G}\widehat{w}$ and $\widehat{w}_t = \widehat{F}\widehat{w}$, and so these two systems are compatible. That is, the compatibility condition

$$\widehat{G}_t - \widehat{F}_x + \widehat{G}\widehat{F} - \widehat{F}\widehat{G} = 0 \tag{4.14}$$

holds. Recall that formulae (2.3)–(2.6) imply the equivalence of the compatibility condition (2.7) to equation (2.14). Taking into account (4.13), put

$$\widehat{\xi} = \begin{bmatrix} 0 & \widehat{v}_1 \\ \widehat{v}_2 & 0 \end{bmatrix} = \xi + i(jX_0j - X_0). \tag{4.15}$$

When equalities (2.4) and (2.5) remain valid after substitution of the matrix functions $\{q_k\}$, $\{Q_k\}$ and ξ by the matrix functions $\{\widehat{q}_k\}$, $\{\widehat{Q}_k\}$ and $\widehat{\xi}$, respectively, then \widehat{G} and \widehat{F} have the same structure as G and F . Therefore, similar to (2.7) equation (4.14) is equivalent to (2.14), that is, \widehat{v}_1 and \widehat{v}_2 satisfy (2.14). According to (4.11) we have

$$\widehat{q}_1 = q_1, \quad \widehat{Q}_2 = Q_2. \tag{4.16}$$

From (4.11) it also follows that

$$2\widehat{q}_0 = -\widehat{Q}_1 = 2(j\widehat{\xi} - i j X_0 + i X_0 j) = 2j(\widehat{\xi} + i(jX_0j - X_0)). \tag{4.17}$$

Substitute (4.15) into (4.17) to get the relation similar to the second relation in (2.4):

$$2\widehat{q}_0 = -\widehat{Q}_1 = 2j\widehat{\xi}. \tag{4.18}$$

It remains to prove that

$$\widehat{Q}_0(x, t) = i(j\widehat{\xi}(x, t)^2 - \widehat{\xi}_x(x, t)). \tag{4.19}$$

By (2.4), (2.5) and (4.12) we obtain

$$\widehat{Q}_0 = i(j\widehat{\xi}^2 - \widehat{\xi}_x) + 2j\widehat{\xi}X_0 - 2X_0j\widehat{\xi} + 2ijY_1 - 2iX_1j + 2iX_0jX_0. \tag{4.20}$$

According to (4.10) we have $A_2S^{-1} = S^{-1}A_1 - S^{-1}\Pi_1\Pi_2^*S^{-1}$. Therefore, in view of (3.11), it follows that

$$Y_1 = \Pi_2^*(S^{-1}A_1 - S^{-1}\Pi_1\Pi_2^*S^{-1})\Pi_1 = X_1 - X_0^2. \tag{4.21}$$

Using (4.21), we rewrite (4.20) as

$$\widehat{Q}_0 = i(j\widehat{\xi}^2 - \widehat{\xi}_x) + 2i(jX_1 - X_1j) - 2ijX_0^2 + 2(j\widehat{\xi}X_0 - X_0j\widehat{\xi}) + 2iX_0jX_0. \tag{4.22}$$

To calculate $-i\widehat{\xi}_x$ note that by (3.11), (3.13), (4.5) and (4.8) we have

$$(X_0)_x = -\widehat{q}_1X_1 - \widehat{q}_0X_0 + X_1q_1 + X_0q_0. \tag{4.23}$$

Taking into account (2.4), (4.16) and (4.18), we rewrite (4.23) as

$$(X_0)_x = -ijX_1 - j\widehat{\xi}X_0 + iX_1j + X_0j\widehat{\xi}. \tag{4.24}$$

By (4.15) and (4.24) we obtain

$$-i\widehat{\xi}_x = -i\widehat{\xi}_x + 2i(jX_1 - X_1j) - \widehat{\xi}X_0j + j\widehat{\xi}X_0 + jX_0j\widehat{\xi}j - X_0j\widehat{\xi}. \tag{4.25}$$

Finally, in view of (4.15), (4.22), (4.25) and equality $\widehat{\xi}j = -j\widehat{\xi}$, a direct calculation shows that

$$\widehat{Q}_0 + i\widehat{\xi}_x = ij\widehat{\xi}^2. \tag{4.26}$$

Thus, equality (4.19) follows. □

Remark 4.2. Note that for the trivial initial solution, that is, for the case $v_1 = 0$ and $v_2 = 0$ the blocks of the matrix functions Π_1 and Π_2 are calculated via (4.3) and (4.5)–(4.7) explicitly

$$\Lambda_s = (\exp(-1)^{s+1} C_1(x, t)) h_s, \quad C_1(x, t) = \frac{i}{2} f(t) (x I_n + a_1)^2, \quad (4.27)$$

$$\Psi_s = (\exp(-1)^{s+1} C_2(x, t)) k_s, \quad C_2(x, t) = \frac{i}{2} \overline{f(t)} (x I_n + a_2^*)^2, \quad (4.28)$$

where h_s and k_s are $n \times m_s$ matrices ($s = 1, 2$), $f(t) = (4(t + b))^{-1}$.

Remark 4.3. Taking into account (4.3) we can rewrite (4.10) in the form

$$a_1 S(x, t) - S(x, t) a_2 = f(t)^{-1} \Pi_1(x, t) \Pi_2(x, t)^*. \quad (4.29)$$

Equation (4.29) is a linear algebraic system from which S is easily recovered. Indeed, let a_1 and a_2 be diagonal matrices

$$a_1 = \text{diag}\{c_1, c_2, \dots, c_n\}, \quad a_2 = \text{diag}\{\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_n\}, \quad \sigma(a_1) \cap \sigma(a_2) = \emptyset, \quad (4.30)$$

where σ is a spectrum, and put

$$R(x, t) = \{R_{kj}(x, t)\}_{k,j=1}^n = f(t)^{-1} \Pi_1(x, t) \Pi_2(x, t)^*. \quad (4.31)$$

Then formulae (4.29) and (4.30) imply

$$S(x, t) = \{S_{kj}(x, t)\}_{k,j=1}^n, \quad S_{kj}(x, t) = (c_k - \tilde{c}_j)^{-1} R_{kj}(x, t). \quad (4.32)$$

That is, taking into account (4.31), we express the entries of S explicitly via Π_1 and Π_2 , which in their turn are constructed explicitly in (4.27) and (4.28). (Recall that $\Pi_1 = [\Lambda_1 \ \Lambda_2]$ and $\Pi_2 = [\Psi_1 \ \Psi_2]$.)

In the general (non-diagonal) situation we denote the columns of S by S_k and the columns of R by R_k and introduce vectors

$$\vec{S} = \begin{bmatrix} S_1 \\ \dots \\ S_n \end{bmatrix}, \quad \vec{R} = \begin{bmatrix} R_1 \\ \dots \\ R_n \end{bmatrix}. \quad (4.33)$$

Next, introduce $n \times n$ block matrices with the blocks of order n , that is $n^2 \times n^2$ matrices,

$$\check{a}_1 = \text{diag}\{a_1, a_1, \dots, a_1\}, \quad \check{a}_2 = \{a_{jk} I_n\}_{k,j=1}^n, \quad (4.34)$$

where a_{jk} are the entries of a_2 . Then, (4.29) yields

$$\vec{S} = (\check{a}_1 - \check{a}_2)^{-1} \vec{R}, \quad (4.35)$$

and S is immediately recovered from \vec{S} . The condition $\sigma(a_1) \cap \sigma(a_2) = \emptyset$ is sufficient for the invertibility of $\check{a}_1 - \check{a}_2$.

Using proposition 4.1 and remarks 4.2 and 4.3, a wide class of solutions of the matrix CNLS with external potential can be constructed explicitly up to the inversion of finite matrices.

Proposition 4.4. Fix arbitrary $n \times n$ matrices a_1 and a_2 such that $\sigma(a_1) \cap \sigma(a_2) = \emptyset$. Let matrices Λ_s and Ψ_s be given by (4.27) and (4.28). Define \vec{S} via (4.31) and (4.33)–(4.35), and recover S columnwise from \vec{S} . Then the matrix functions

$$\hat{v}_1 = -2i \Psi_1^* S^{-1} \Lambda_2, \quad \hat{v}_2 = -2i \Psi_2^* S^{-1} \Lambda_1 \quad (4.36)$$

satisfy CNLS (2.14).

Example 4.5. Assume $n = 1$ (i.e., A_1 and A_2 are now scalar functions) and $a_1 \neq a_2$. Then the recovery of $S(x, t)$ is especially simple. Using (4.10), we obtain

$$S(x, t) = (A_1(x, t) - A_2(x, t))^{-1} (\Lambda_1(x, t)\Psi_1(x, t)^* + \Lambda_2(x, t)\Psi_2(x, t)^*). \tag{4.37}$$

By (4.3), (4.27) and (4.28) we rewrite (4.37) as

$$S(x, t) = \frac{1}{f(t)(a_1 - a_2)} \left(\exp \left\{ \frac{i}{2} f(t) \left(2x(a_1 - a_2) + a_1^2 - a_2^2 \right) \right\} h_1 k_1^* + \exp \left\{ \frac{i}{2} f(t) \left(2x(a_2 - a_1) + a_2^2 - a_1^2 \right) \right\} h_2 k_2^* \right). \tag{4.38}$$

From (4.13), (4.27) and (4.28) it follows that

$$\widehat{v}_1 = -\frac{2i}{S(x, t)} \exp \left\{ -\frac{i}{2} f(t) ((x + a_1)^2 + (x + a_2)^2) \right\} k_1^* h_2, \tag{4.39}$$

$$\widehat{v}_2 = -\frac{2i}{S(x, t)} \exp \left\{ \frac{i}{2} f(t) ((x + a_1)^2 + (x + a_2)^2) \right\} k_2^* h_1, \tag{4.40}$$

where S is given by (4.38).

Now, let us construct GBDT for the KdV-type equation (2.24). We shall construct self-adjoint solutions. For simplicity we shall also put the initial solution $v = 0$. (This assumption is often made, when explicit solutions are constructed.) Partition $\Pi_1(x, t)$ into two $p \times p$ blocks: $\Pi_1 = [\Lambda_1 \quad \Lambda_2]$. The coefficients in (3.5) and (3.6) for the cases $u = x$ and $u = t$ are given by equalities (2.19) and (2.21), respectively, after substitution $v = 0$. In other words, the first relation in (3.5) can be rewritten as

$$\frac{\partial}{\partial x} \Lambda_1 = \alpha \Lambda_2, \quad \frac{\partial}{\partial x} \Lambda_2 = -\Lambda_1, \tag{4.41}$$

$$\frac{\partial}{\partial t} \Lambda_1 = g(4\alpha^2 \Lambda_2 + 2f \Lambda_1), \quad \frac{\partial}{\partial t} \Lambda_2 = -g(4\alpha \Lambda_1 + 2f \Lambda_2), \tag{4.42}$$

where $\alpha = A_1$. According to (2.23) formula (3.4) ($l = 1$) takes the form

$$\alpha_x = f I_n, \quad \alpha_t = 12fg\alpha. \tag{4.43}$$

Suppose

$$g(t) = \overline{g(t)}, \quad f(t) = \overline{f(t)}. \tag{4.44}$$

Taking into account (4.41)–(4.44) it is easy to check, that A_2 and Π_2 given by

$$\Pi_2 = \Pi_1 J^*, \quad A_2 = \alpha^*, \quad J = \begin{bmatrix} 0 & I_p \\ -I_p & 0 \end{bmatrix}, \tag{4.45}$$

satisfy relation (3.4) ($l = 2$) and the second relation in (3.5) for $u = x$ and $u = t$. In view of (2.23) relations (3.6) take the form

$$S_x = \Lambda_2 \Lambda_2^*, \quad S_t = 4g(\alpha \Lambda_2 \Lambda_2^* + \Lambda_2 \Lambda_2^* \alpha^* + \Lambda_1 \Lambda_1^*) - 12fgS. \tag{4.46}$$

Finally, relations (3.13) take the form

$$(\Pi_2^* S^{-1})_x = -\sum_{k=0}^1 \widehat{q}_k \Pi_2^* S^{-1} \alpha^k, \quad (\Pi_2^* S^{-1})_t = 12fg \Pi_2^* S^{-1} - \sum_{k=0}^2 \widehat{Q}_k \Pi_2^* S^{-1} \alpha^k, \tag{4.47}$$

where coefficients \widehat{q}_k and \widehat{Q}_k are obtained via transformation (3.10). Equality (3.3) can be written as

$$\alpha(0, 0)S(0, 0) - S(0, 0)\alpha(0, 0)^* = \Lambda(0, 0)J\Lambda(0, 0)^* \quad \Lambda := \Pi_1. \tag{4.48}$$

It yields the identity

$$\alpha(x, t)S(x, t) - S(x, t)\alpha(x, t)^* = \Lambda(x, t)J\Lambda(x, t)^*. \tag{4.49}$$

Proposition 4.6. *Let relations (2.23), (4.41)–(4.44), (4.46) and (4.48) hold, and let $\det S \neq 0$. Put*

$$\widehat{v} = 2(\Omega_{12} + \Omega_{21} + \Omega_{22}^2), \quad \Omega_{kj} := \Lambda_k^* S^{-1} \Lambda_j. \tag{4.50}$$

Then, in the points of invertibility of S , the matrix function \widehat{v} is a self-adjoint solution of the KdV-type equation (2.24).

Proof. Similar to the isospectral KdV case, transformation (3.10) of the coefficients q_k and Q_k does not preserve their structure. Therefore we cannot use theorem 3.2 in this proof, and so we will show that \widehat{v} satisfies (2.24) directly, using (3.13), i.e., formulae (4.47). The GBDT solution of the isospectral classical KdV equation was treated in detail in [16], section 5. Relations (4.41) and the first relation in (4.46) coincide with formula (5.6) in [16]. By these relations we get

$$\frac{\partial}{\partial x} \Omega_{12} = -\Omega_{12}\Omega_{22} - \Omega_{11} + \Lambda_2^* \alpha^* S^{-1} \Lambda_2, \quad \frac{\partial}{\partial x} \Omega_{21} = -\Omega_{22}\Omega_{21} - \Omega_{11} + \Lambda_2^* S^{-1} \alpha \Lambda_2, \tag{4.51}$$

$$\frac{\partial}{\partial x} \Omega_{22} = -(\Omega_{22}^2 + \Omega_{12} + \Omega_{21}) = -\frac{1}{2}\widehat{v}, \quad \frac{\partial}{\partial x} \Omega_{22}^2 = -\frac{1}{2}(\widehat{v}\Omega_{22} + \Omega_{22}\widehat{v}), \tag{4.52}$$

$$\frac{\partial}{\partial x} \Omega_{11} = -\Omega_{12}\Omega_{21} + \Lambda_2^* \alpha^* S^{-1} \Lambda_1 + \Lambda_1^* S^{-1} \alpha \Lambda_2. \tag{4.53}$$

After proper change of notation, definition (4.50) coincides with formula (5.16) in [16]. Using relations (4.41), (4.47) and (4.49)–(4.53) similar to [16] we obtain derivatives \widehat{v}_{xx} and \widehat{v}_{xxx} . Some additional terms appear as α depends now on x , i.e., $\alpha_x = f(t)I_p$, whereas we had $\alpha \equiv \text{const}$ in the isospectral case [16]. In particular, we have

$$3\widehat{v}^2 - \widehat{v}_{xx} = 8(\Omega_{21}\Omega_{12} + \Lambda_2^* S^{-1} \alpha \Lambda_2 \Omega_{22} + \Omega_{22} \Lambda_2^* \alpha^* S^{-1} \Lambda_2 + \Lambda_2^* S^{-1} \alpha \Lambda_1 + \Lambda_1^* \alpha^* S^{-1} \Lambda_2) - 4f\Omega_{22}, \tag{4.54}$$

where $-4f\Omega_{22}$ is such an additional term (compare with formula (5.31) in [16]). After differentiation of the right-hand side of (4.54) we obtain

$$(3\widehat{v}^2 - \widehat{v}_{xx})_x = R_1 + R_2, \tag{4.55}$$

where

$$R_1 = 8(\Lambda_2^* (\alpha^*)^2 S^{-1} \Lambda_2 + \Lambda_2^* S^{-1} \alpha^2 \Lambda_2 - (\Lambda_1^* S^{-1} \alpha \Lambda_1 + \Lambda_1^* \alpha^* S^{-1} \Lambda_1 + (\Lambda_1^* S^{-1} \alpha \Lambda_2 + \Lambda_1^* \alpha^* S^{-1} \Lambda_2 + \Lambda_2^* S^{-1} \alpha \Lambda_1) \Omega_{22} + \Lambda_2^* S^{-1} \alpha \Lambda_2 \Omega_{21} + \Omega_{12} \Lambda_2^* \alpha^* S^{-1} \Lambda_2 + \Omega_{11} \Omega_{12} + \Omega_{21} \Omega_{11} + \Omega_{22} (\Lambda_2^* \alpha^* S^{-1} \Lambda_1 + \Lambda_1^* \alpha^* S^{-1} \Lambda_2 + \Lambda_2^* S^{-1} \alpha \Lambda_1) + \Omega_{22} (\Lambda_2^* S^{-1} \alpha \Lambda_2 \Omega_{22} + \Omega_{22} \Lambda_2^* \alpha^* S^{-1} \Lambda_2 + \Omega_{21} \Omega_{12})) + (\Lambda_2^* S^{-1} \alpha \Lambda_2 \Omega_{22} + \Omega_{22} \Lambda_2^* \alpha^* S^{-1} \Lambda_2 + \Omega_{21} \Omega_{12}) \Omega_{22}), \tag{4.56}$$

and R_2 is the additional, with respect to the isospectral case, term:

$$R_2 = 16f\Omega_{22}^2 + 8f(\Omega_{12} + \Omega_{21}) - 4f(\Omega_{22})_x.$$

In view of (4.50) and (4.52) we have

$$R_2 = 8f\Omega_{22}^2 + 6f\widehat{v}. \tag{4.57}$$

Now, consider \widehat{v}_t . By (2.23), (3.10) and (4.45) we get

$$\begin{aligned} \widehat{Q}_2 &= Q_2, & \widehat{Q}_1 &= Q_1 - Q_2 J \Pi_1^* S^{-1} \Pi_1 + J \Pi_1^* S^{-1} \Pi_1 Q_2, \\ \widehat{Q}_0 &= Q_0 - Q_1 J \Pi_1^* S^{-1} \Pi_1 + J \Pi_1^* S^{-1} \Pi_1 Q_1 \\ &\quad - Q_2 J \Pi_1^* \alpha^* S^{-1} \Pi_1 + J \Pi_1^* S^{-1} \alpha \Pi_1 Q_2 - J \Pi_1^* S^{-1} \Pi_1 Q_2 J \Pi_1^* S^{-1} \Pi_1. \end{aligned} \quad (4.58)$$

From (4.42), the second equality in (4.47), (4.58) and identity (4.49) it follows that

$$\begin{aligned} \frac{\partial}{\partial t} \Omega_{12} &= 4g(\Lambda_2^* (\alpha^*)^2 S^{-1} \Lambda_2 - \Lambda_1^* S^{-1} (\alpha \Lambda_2 \Lambda_2^* \\ &\quad + \Lambda_2 \Lambda_2^* \alpha^* + \Lambda_1 \Lambda_1^*) S^{-1} \Lambda_2 - \Lambda_1^* S^{-1} \alpha \Lambda_1) + 12gf \Omega_{12}, \end{aligned} \quad (4.59)$$

$$\begin{aligned} \frac{\partial}{\partial t} \Omega_{21} &= 4g(\Lambda_2^* S^{-1} \alpha^2 \Lambda_2 - \Lambda_2^* S^{-1} (\alpha \Lambda_2 \Lambda_2^* \\ &\quad + \Lambda_2 \Lambda_2^* \alpha^* + \Lambda_1 \Lambda_1^*) S^{-1} \Lambda_1 - \Lambda_1^* \alpha^* S^{-1} \Lambda_1) + 12gf \Omega_{21}, \end{aligned} \quad (4.60)$$

$$\begin{aligned} \frac{\partial}{\partial t} \Omega_{22} &= -4g(\Lambda_1^* \alpha^* S^{-1} \Lambda_2 + \Lambda_2^* S^{-1} (\alpha \Lambda_2 \Lambda_2^* \\ &\quad + \Lambda_2 \Lambda_2^* \alpha^* + \Lambda_1 \Lambda_1^*) S^{-1} \Lambda_2 + \Lambda_2^* S^{-1} \alpha \Lambda_1) + 8gf \Omega_{22}. \end{aligned} \quad (4.61)$$

Taking into account (4.50), (4.56) and (4.59)–(4.61) we derive

$$\widehat{v}_t = gR_1 + 12gf\widehat{v} + 8gf\Omega_{22}^2. \quad (4.62)$$

Finally, compare formulae (4.55), (4.57) and (4.62) to get

$$\widehat{v}_t - 6gf\widehat{v} = g(3\widehat{v}^2 - \widehat{v}_{xx})_x. \quad \square$$

Remark 4.7. To construct Λ_1 and Λ_2 , take into account $f_t = 12gf^2$ and note that the matrix function α of the form $\alpha(x, t) = f(t)(xI_n + a)$, where a is an $n \times n$ matrix, satisfies (4.43). It follows from (2.7) and can also be checked directly that

$$\frac{\partial^2}{\partial x \partial t} (\Lambda_s) = \frac{\partial^2}{\partial t \partial x} (\Lambda_s) \quad (s = 1, 2).$$

Thus, systems (4.41) and (4.42) are compatible. Using these systems and expression for α , we can recover Λ from the relations

$$\begin{bmatrix} \Lambda_1(0, t) \\ \Lambda_2(0, t) \end{bmatrix} = \exp\{\beta(t)\} \begin{bmatrix} \Lambda_1(0, 0) \\ \Lambda_2(0, 0) \end{bmatrix}, \quad (4.63)$$

$$\begin{bmatrix} \Lambda_1(x, t) \\ \Lambda_2(x, t) \end{bmatrix} = \exp\{\gamma(x, t)\} \begin{bmatrix} \Lambda_1(0, t) \\ \Lambda_2(0, t) \end{bmatrix}, \quad (4.64)$$

where

$$\beta(t) = \frac{1}{3}(f(t) - f(0)) \begin{bmatrix} 0 & a^2 \\ 0 & 0 \end{bmatrix} + 2 \left(\int_0^t g(u) f(u) du \right) \begin{bmatrix} I_p & 0 \\ -2a & -I_p \end{bmatrix}, \quad (4.65)$$

$$\gamma(x, t) = \begin{bmatrix} 0 & \frac{1}{2} f(t) x (x I_p + 2a) \\ -x I_p & 0 \end{bmatrix}. \quad (4.66)$$

Function f is recovered from g by the formula $f(t) = 1/(-12\partial_t^{-1}g)$.

Remark 4.8. As $\alpha(x, t) = f(t)(xI_n + a)$, we can rewrite (4.49) in the form

$$aS(x, t) - S(x, t)a^* = R(x, t), \quad R(x, t) = f(t)^{-1}\Lambda(x, t)J\Lambda(x, t)^*, \quad (4.67)$$

$\Lambda = [\Lambda_1 \ \Lambda_2]$. Using (4.67), $S(x, t)$ is recovered from $R(x, t)$ similar to the way, in which S is recovered from R in remark 4.3.

By proposition 4.6 a class of the self-adjoint solutions of the KdV-type equation (2.24) can be constructed explicitly up to the inversion of finite matrices.

Corollary 4.9. Fix arbitrary $n \times n$ matrix a such that $\sigma(a) \cap \sigma(a^*) = \emptyset$. Let matrices $\Lambda_s (s = 1, 2)$ be given by (4.63)–(4.66). Use remark 4.8 to recover S from $\Lambda_s (s = 1, 2)$. Then the matrix function \widehat{v} , which is defined in (4.50) via S and $\Lambda_s (s = 1, 2)$, satisfies (2.24).

5. Conclusion

Thus, auxiliary systems for matrix nonisospectral equations were introduced, and the GBDT version of the Bäcklund–Darboux transformation was applied. It proved fruitful for the construction of the explicit solutions of the nonisospectral equations, including matrix equations and equations with variable coefficients. In our next work we plan to consider examples with non-diagonal parameter matrices $A(x, t)$ in greater detail.

A comparison of the isospectral and nonisospectral solutions could be fruitful. Note that equality (4.13) for the solution of the nonisospectral CNLS formally coincides with equality (3.13) [29] for the solution of the isospectral CNLS, and equality (4.50) for the solution of the nonisospectral KdV formally coincides with the equality (5.4) [16] for the solution of the isospectral KdV. However, the dependence on x and t and asymptotic behavior of the terms on the right-hand sides of the equality (4.13) in our paper and (3.13) in [29] is quite different. The same is true for the right-hand sides of (4.50) and (5.4) [16]. In particular, assume that $\Im b < 0, k_1 \neq 0, h_2 \neq 0$ in example 4.5. Then we have

$$\lim_{|x| \rightarrow \infty} \|\widehat{v}_1(x, t)\| = \infty, \quad \lim_{|x| \rightarrow \infty} \|\widehat{v}_2(x, t)\| = 0.$$

Nevertheless, an asymmetric soliton was found [4] in the limit $t \rightarrow \infty$ for a solution of the Maxwell–Bloch system with damping. Such analogies are of special interest and we are planning to look for them in our next work. Finally, note that the poles of the Darboux matrix w_A of the form (3.8) are defined by the equality $\det(A_1 - \lambda I_n) = 0$. In the isospectral case the corresponding values of the spectral parameter do not depend on x and t . In view of (2.13) and (4.3) we see that for the nonisospectral CNLS (2.14) the equality $\det(A_1 - \lambda I_n) = 0$ is equivalent to the equality $\det(a_1 - c I_n) = 0$. Here we can talk about the poles of the Darboux matrix with respect to the ‘hidden’ spectral parameter treated in [5].

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